CALCULUS BC

9.3 – Parametric Equations and Calculus

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$,

then the slope of C at the point (x, y) is given by $\boxed{\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}}$ where $\dfrac{dx}{dt} \neq 0$,

Memorize

and the second derivative is given by $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] = \boxed{\dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}}$.

Ex. 1 (Noncalculator)

Given the parametric equations $x = t^2 + 9$ and $y = 4t^5 - 6t^3 + 3$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

$x(t) = t^2 + 9 \qquad y(t) = 4t^5 - 6t^3 + 3$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{20t^4 - 18t^2}{2t} = \frac{20t^4}{2t} - \frac{18t^2}{2t} = 10t^3 - 9t$$ ← slope of parametric graph

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(10t^3 - 9t)}{2t} = \frac{30t^2 - 9}{2t}$$

Ex. 2 (Noncalculator)

Given the parametric equations $x = 5\sin t$ and $y = 3\cos t$, write an equation of the tangent line to the curve at the point where $t = \dfrac{2\pi}{3}$.

Point: $x\left(\frac{2\pi}{3}\right) = 5\sin\left(\frac{2\pi}{3}\right) = 5 \cdot \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$

$y\left(\frac{2\pi}{3}\right) = 3\cos\left(\frac{2\pi}{3}\right) = 3 \cdot -\frac{1}{2} = -\frac{3}{2}$

Pnt: $\left(\frac{5\sqrt{3}}{2}, -\frac{3}{2}\right)$

Slope: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-3\sin t}{5\cos t} = -\frac{3}{5}\tan t \Big|_{t=\frac{2\pi}{3}}$

$= -\frac{3}{5}\tan\left(\frac{2\pi}{3}\right) = -\frac{3}{5} \cdot -\sqrt{3} = \frac{3\sqrt{3}}{5}$

Equation: $y + \frac{3}{2} = \frac{3\sqrt{3}}{5}\left(x - \frac{5\sqrt{3}}{2}\right)$

Ex 3 (Noncalculator)

→ numerator of derivative = 0

Find all points of horizontal and vertical tangency given the parametric equations $x = 3t^2 - 6t$, $y = t^2 - 8t + 9$.

→ denominator of derivative = 0

Horizontal: $\dfrac{dy}{dt} = 0$

$2t - 8 = 0$

$2t = 8$

$t = 4$

$x(4) = 3(4)^2 - 6(4)$

$= 48 - 24$

$= 24$

$y(4) = (4)^2 - 8(4) + 9$

$= 16 - 32 + 9$

$= -7$

Horizontal tangency occurs at the point $(24, -7)$

Vertical: $\dfrac{dx}{dt} = 0$

$6t - 6 = 0$

$6t = 6$

$t = 1$

$x(1) = 3(1)^2 - 6(1)$

$= 3 - 6$

$= -3$

$y(1) = (1)^2 - 8(1) + 9$

$= 1 - 8 + 9$

$= 2$

Vertical tangent occurs at the point $(-3, 2)$

Earlier in the year we learned to find the arc length of a curve C given by $y = h(x)$ over the interval $x_1 \le x \le x_2$ by using the formula $s = \int_{x_1}^{x_2} \sqrt{1 + (h'(x))^2}\, dx = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$

If C is represented by the parametric equations $x = f(t)$ and $y = g(t)$ over the interval $a \le t \le b$, then

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, dx = \int_{x_1}^{x_2} \sqrt{1 + \frac{(dy/dt)^2}{(dx/dt)^2}}\, dx$$

$$= \int_{x_1}^{x_2} \sqrt{\frac{(dx/dt)^2}{(dx/dt)^2} + \frac{(dy/dt)^2}{(dx/dt)^2}}\, dx = \int_{x_1}^{x_2} \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\, dx$$

$$= \int_{x_1}^{x_2} \frac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{dx/dt}\, dx = \int_{x_1}^{x_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dx \cdot \frac{dt}{dx}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \leftarrow$$ change terms from $x \to t$ so change bounds

Length of arc for parametric graphs is $\boxed{s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$. Memorize

Also used to find... Magnitude of velocity vector (speed) of particle along curve

Note that the formula works when the curve does not intersect itself on the interval $a \le t \le b$, and the curve must be smooth.

Ex. 4 (Noncalculator)
Set up an integral expression for the arc length of the curve given by the parametric equations $x = e^{5t+2}$, $y = 3\sin(t^2)$, $0 \le t \le 4$. Do not evaluate.

$$s = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^4 \sqrt{(5e^{5t+2})^2 + (6t\cos(t^2))^2}\, dt$$

Ex. Which of the following gives the length of the path described by the parametric equations $x=\sin(t^3)$ and $y=e^{5t}$ from $t = 0$ to $t = \pi$?

$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

(A) $\int_0^\pi \sqrt{\sin^2(t^3)+e^{10t}}\,dt$ (B) $\int_0^\pi \sqrt{\cos^2(t^3)+e^{10t}}\,dt$ (C) $\int_0^\pi \sqrt{9t^4\cos^2(t^3)+25e^{10t}}\,dt$

(D) $\int_0^\pi \sqrt{3t^2\cos(t^3)+5e^{5t}}\,dt$ (E) $\int_0^\pi \sqrt{\cos^2(t^3)+e^{10t}}\,dt$

$\left(\frac{dx}{dt}\right)^2 = [3t^2\cos(t^3)]^2 = 9t^4\cos^2(t^3)$

$\left(\frac{dy}{dt}\right)^2 = [5e^{5t}]^2 = 25e^{10t}$

Ex. A curve C is defined by the parametric equations $x=t^2-4t+1$ and $y=t^3$. Which of the following is an equation of the line tangent to the graph of C at the point $(-3, 8)$?

(A) $x=-3$ (B) $x=2$ (C) $y=8$ (D) $y=-\frac{27}{10}(x+3)+8$ (E) $y=12(x+3)+8$

Point: $(-3, 8)$

slope: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t-4}$ ← need to find t

$8 = t^3$

$\sqrt[3]{8} = t$

$t = 2$

$= \frac{3(2)^2}{2(2)-4} = \frac{3(4)}{4-4}$ = undefined : vertical = x value

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Ex. The length of the path described by the parametric equations $x=\frac{1}{3}t^3$ and $y=\frac{1}{2}t^2$, where $0 \le t \le 1$, is given by

$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

(A) $\int_0^1 \sqrt{t^2+1}\,dt$ (B) $\int_0^1 \sqrt{t^2+t}\,dt$ (C) $\int_0^1 \sqrt{t^4+t^2}\,dt$ (D) $\frac{1}{2}\int_0^1 \sqrt{4+t^4}\,dt$ (E) $\frac{1}{6}\int_0^1 t^2\sqrt{4t^2+9}\,dt$

$\left(\frac{dx}{dt}\right)^2 = (t^2)^2 = t^4$

$\left(\frac{dy}{dt}\right)^2 = (t)^2 = t^2$

Ex. A particle moves in the xy-plane so that its position for $t \geq 0$ is given by the parametric equations $x = \ln(t+1)$ and $y = kt^2$, where k is a positive constant. The line tangent to the particle's path at the point where $t = 3$ has slope 8. What is the value of k?

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2kt}{\frac{1}{t+1}} = 2kt(t+1)\Big|_{t=3}$$

$$= 2k(3)\cdot(3+1) = 6k(4) = 24k$$

$$24k = 8$$
$$k = \frac{1}{3}$$

Ex. A curve is defined by the parametric equations $x(t) = 3e^{2t}$ and $y(t) = e^{3t} - 1$. What is $\frac{d^2y}{dx^2}$ in terms of t?

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3e^{3t}}{6e^{2t}} = \frac{1}{2}e^{t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{1}{2}e^{t}\right]}{6e^{2t}} = \frac{\frac{1}{2}e^{t}}{6e^{2t}} = \frac{1}{12e^{t}}$$